A Theoretical Model for Mutual Interaction between Coaxial Cylindrical Coils

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Abstract: The wireless power transfer link between two coils is determined by the properties of the coils and their mutual interaction. A theoretical model, based on classical electrodynamics, is developed to describe the interaction between coaxial cylindrical coils at low frequencies. Therefore, vector potentials and symmetry arguments are used to solve Maxwell's equations in the quasi-static limit. Expressions for the mutual inductance, coil-resistance due to skin effects and proximity effects are derived.

Keywords: Mutual inductance, coupling factor, skin effect, proximity effects, vector potentials

Physical quantities

μ ....Magnetic permeability
σ ....Conductivity
ε ....Electric permittivity
$\boldsymbol{E}$ ....Electric field
$\boldsymbol{B}$ ....Magnetic field
$\boldsymbol{A}$ ....Vector potential
$\boldsymbol{J}$ ....Current density
I ....Current
r ....Filament radius
h........Filament separation
δ ....Skin depth
ω ....Angular frequency

t .... Time
L .... Self-inductance
M .... Mutual inductance
V .... Potential difference
R .... Resistance
k .... Coupling factor
l .... Conductor length
d .... Conductor diameter
ξ .... Surface charge density
$\boldsymbol{K}$ .... Surface current density

Bold letters are vectors

Mathematical quantities

$\boldsymbol{e}$ ....Unit vector
i ....Complex number
$\delta(x)$ ...Dirac delta function
dl ....Infinitesimal line element
J_1 ....First kind Bessel function of first order
$\boldsymbol{\nabla}$ ....Nabla operator
$\wedge$ ....Vector cross product
∇^2 ....Laplace operator
$\boldsymbol{\nabla}^2$ ....Vector Laplace operator
$\mathbb{R}_+$ ....Real numbers $[0, \infty)$

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1 Introduction

Recent treatments of wireless power transfer between coils [refA] have shown that understanding the mutual interaction is a major task when optimizing the link efficiency. Here, we give a theoretical approach on the topic of mutual interaction and resistive losses due to skin- and proximity effects. The derivation of the theoretical model is split into three parts. First, the mutual interaction between two infinitesimal thin coaxial circular filaments is established. In the second step, these infinitesimal thin filaments are extended to filaments of finite thickness. The main idea by this part is to regard skin- and proximity effects. Finally, the behavior of coaxial circular coils, having multiple turns and layers, is derived from the first to parts by the principle of superposition.
Some general assumptions, valid for the entire article, have to be mentioned. All media used for the theoretical models are assumed to be homogeneous, isotropic and linear. Moreover, any field or current that varies in time changes slowly enough, such that the quasi static limit is valid and the electric permittivity ε, magnetic permeability μ and electric conductivity σ are constant in every medium.

2 Two coaxial circular filaments

2.1 Mutual inductance

We start with two coaxial circular filaments in free space, see Figure 1. Each filament is represented by a harmonically varying current density distribution.

$$\boldsymbol{J}_{1,2} = I_{1,2}\delta(z)\frac{\delta(\rho - r_{1,2})}{r_{1,2}}\boldsymbol{e}_\phi e^{i\omega t}$$

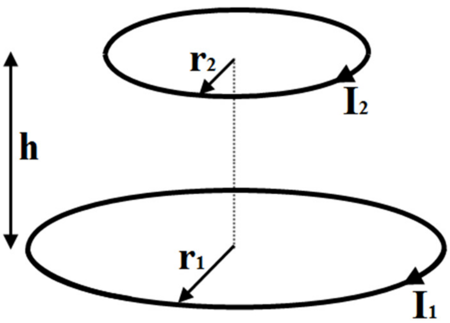

Figure 1: Two coaxial circular filaments in free space

The vector potential as a function of radius and height in a cylindrical coordinate frame, resulting from $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ (derivation given in the appendix) is

$$\boldsymbol{A}(\rho, z) = \frac{\mu}{2}\int_0^\infty dm\left[r_1 I_1 J_1(mr_1)e^{-m|z|} + r_2 I_2 J_1(mr_2)e^{-m|z-h|}\right]J_1(m\rho)\boldsymbol{e}_\phi. \tag{1}$$

Apparently, the first term inside the integral of equation (1) corresponds to a vector potential induced by $\boldsymbol{J}_1$ and the second term to a potential induced by $\boldsymbol{J}_2$. For filament one, the first term of (1) is denoted as the self-induced potential and the second term as the mutual-induced potential. Hence, there are two expressions for the self- and mutual-vector potential:

$$\boldsymbol{A_L}(\rho, z) = \frac{\mu}{2}\int_0^\infty dm\, r_1 I_1 J_1(mr_1)J_1(m\rho)e^{-m|z|}\boldsymbol{e}_\phi \tag{2}$$

$$\boldsymbol{A_M}(\rho, z) = \frac{\mu}{2}\int_0^\infty dm\, r_2 I_2 J_1(mr_2)J_1(m\rho)e^{-m|z-h|}\boldsymbol{e}_\phi \tag{3}$$

According to [refB], the self-induction L_1 and mutual induction M_{21} for loop one are given by

$$L_1 = \frac{1}{I_1}\oint_{\partial C_1} \boldsymbol{A}_L . d\boldsymbol{l}, \tag{4}$$

$$M_{21} = \frac{1}{I_2}\oint_{\partial C_1} \boldsymbol{A}_M . d\boldsymbol{l}, \tag{5}$$

where ∂C_1 denotes filament one and $d\boldsymbol{l} = dl\,\boldsymbol{e}_\phi$ an infinitesimal tangential vector line element. Thus, we can get integral solutions for the self-inductance

$$L_1 = \pi\mu r_1^2 \int_0^\infty dm\, J_1^2(mr_1), \tag{6}$$

and for the mutual-inductance

$$M_{21} = \pi\mu r_1 r_2 \int_0^\infty dm\, J_1(mr_1) J_1(mr_2) e^{-mh}. \tag{7}$$

Similarly, expression for L_2 and M_{12} are:

$$L_2 = \pi\mu r_2^2 \int_0^\infty dm\, J_1^2(mr_2) \tag{8}$$

$$M_{12} = M_{21} \equiv M \tag{9}$$

Note that the self-inductances $L_1$and L_2 cannot be computed because the integral expressions diverge. This is due to the infinitesimal thickness of the filaments.

2.2 Coupling factor

The coupling factor between two coils is defined by:

$$k = \frac{M}{\sqrt{L_1 L_2}} \tag{10}$$

Using the relations for self- and mutual-inductance, one can find that $0 \leq k \leq 1$. Furthermore, the self-inductances L_1 and L_2 do not depend on the coil separation h, whereas the mutual inductance does. For a small gap h, approximate $e^{-mh} \approx 1 - mh$ and

$$k \approx \frac{\pi\mu r_1 r_2}{\sqrt{L_1 L_2}} \int_0^\infty dm\, J_1(mr_1) J_1(mr_1)(1 - mh) = c - c'h, \tag{11}$$

where c and c' are constants, independent on h. Equation (16) shows, that the coupling factor falls off linearly in case of small separation. Experimental measurements agree with this, e.g. see [refA] or [refC].

3 Filaments of finite thickness

3.1 Regarding eddy currents

Many examples like [refA] show that the coil resistance increases significantly with frequency. This is due to the fact, that harmonically time-varying electromagnetic fields near a conductor induce so called eddy currents. These eddy currents are frequency dependent and affect the current distribution within the conductor. If the electromagnetic field is produced by a current distribution inside the conductor, the phenomenon is called skin effect. But if the electromagnetic field outside the conductor is produced by other coils or current distribution, the change of the effective resistance is called proximity effect. In order to find a model for the AC resistance, it is necessary to regard these eddy currents. Therefore, the diffusion of the vector potential near a conducting surface has to be investigated. For a first order approximation, consider a local vector potential field $\boldsymbol{A}_0 = A_0\boldsymbol{e}_x$ parallel to a conducting surface. For the mathematical calculations, use a Cartesian reference frame where the surface conductor is set to the xy plane, see Figure 2.

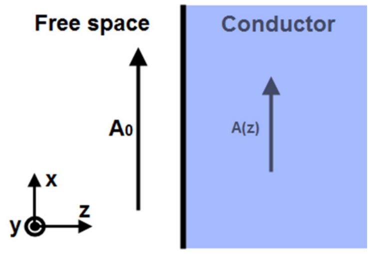

Figure 2: Boundary between free space and a conductor

The setup is basically the same as given in [refB], but for vector potentials. From the boundary conditions, the parallel component of $\boldsymbol{A_0}$ is conserved and the dependence of the vector potential inside the conductor gets $\boldsymbol{A} = A(z)\boldsymbol{e}_x$.The fact that $A(z)$ is independent on the x-y coordinate and only has a component in x direction, the diffusion equation for the vector potential reduces to $\boldsymbol{\nabla A} - i\omega\mu\sigma\boldsymbol{A} = 0$. Hence,

$$\frac{d^2}{dz^2}A(z) - i\omega\mu\sigma A(z) = 0 \tag{12}$$

A solution to differential equation (12) is

$$A(z) = ce^{ikz} + c'e^{-ikz} \tag{13}$$

Where $k = (1+i)\sqrt{\frac{\omega\mu\sigma}{2}}$. By definition, the skin depth δ is $\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$. The physically relevant solution of (13) can be written as

$$A(z) = A_0 e^{-z/\delta} e^{iz/\delta} \tag{14}$$

Note that the coefficient c' was determined by the boundary condition at $z = 0$. From the definition of $\boldsymbol{E}$ and by Ohm's law, the current density $\boldsymbol{J}$ inside the conductor is

$$J(z) = -i\omega\sigma A_0 e^{-z/\delta} e^{iz/\delta} \tag{15}$$

Equation (15) states that the current density falls off exponentially from the surface of the conductor.

3.2 Resistance for a cylindrical conductor

Consider a cylindrical conductor in free space with a vector potential $\boldsymbol{A_0}$ outside (see Figure 4). At high frequencies, $\delta \ll d$ holds. To handle the exponential decay, $J(z)$ is approximated by a formal current density $\hat{J}(z)$:

$$\hat{J}(z) \equiv \begin{cases} \omega\sigma A_0, & 0 < z < \delta \\ 0, & z \geq \delta \end{cases} \tag{16}$$

This means, that the current flows only through a shell of thickness δ at the boundary of the conductor. It can be shown, that both current densities $J(z)$and $\hat{J}(z)$ carry the same amount of current for a cylindrical conductor of diameter d, when $d \gg \delta$. The total current I

through the conductor is given by the integral of the current density over the conductor cross-section, namely:

$$I = \int J\, da = 2\pi \int_0^{d/2} zdz\, J(z) \qquad \hat{I} = \int \hat{J}\, da = 2\pi \int_0^{d/2} zdz\, \hat{J}(z) \tag{17}$$

The result of the integration is:

$$I = \pi\omega\sigma A_0\delta\left(\delta + \left((i-1)\frac{d}{2} - \delta\right)e^{-d/2\delta}e^{id/2\delta}\right) \qquad \hat{I} = \pi\omega\sigma A_0\delta^2 \tag{18}$$

$$I \approx \pi\omega\sigma A_0\delta^2, \text{for } d \gg \delta$$

Both current densities carry the same current. For simplicity, $\hat{I}$ is used for further calculations. Thus, a first order approximation assumes that the current inside a conductor flows through a shell of radius δ at the boundary of the conductor. This fact also changes the resistance of the conductor since the effective cross-section is reduced. Consider a circular conductor with cross-section diameter d. A potential difference V over a length l produces an electric field E. This potential difference can be expressed by the resistance R and a current I through the conductor. This gives $E = \frac{V}{l} = \frac{R\,I}{l}$. By the electric field, a current density $J = \sigma E$ is induced. The total current through the shell $\hat{I}$ (see equation (25)) is equal to $\int J\, da$, i.e.

$$\hat{I} = \int J\, da = \sigma\frac{R\,I}{l}a \tag{19}$$

Here, a is the cross-section area for a shell of thickness δ. For cylindrical conductors with diameter d:

$$a = \pi\left(\frac{d}{2}\right)^2 - \pi\left(\frac{d}{2} - \delta\right)^2 = \pi d\delta - \delta^2 \approx \pi d\delta \qquad \text{for } d \gg \delta. \tag{20}$$

Skin effect: The vector potential outside can be described by the self-inductance, namely

$$A_0 = \frac{IL}{l}. \tag{21}$$

Hence, the resistance for the skin effect is

$$R_{skin} \approx \frac{2}{\mu\sigma\delta d}L \tag{22}$$

Proximity effects: According to (7), A_0 can be described by the mutual inductance and the current I_2 through the external loop.

$$A_0 = \frac{I_2 M}{l} \tag{23}$$

The resistance for proximity effects is then

$$R_{prox} \approx \frac{2}{\mu\sigma\delta d}\frac{I_2}{I_1}M. \tag{24}$$

Finally, the total resistance of the current loops for finite thickness is

$$R = R_{skin} + R_{prox} \approx \frac{2}{\mu\sigma\delta d}\left(L + \frac{I_2}{I_1}M\right). \tag{25}$$

4 Cylindrical coils

4.1 Principle of linear superposition

Since Maxwell's equations are linear, the diffusion equation for the vector potential derived in the appendix is also linear. Hence, the principle of superposition can be used for more than two circular filaments in free space.

4.2 Mutual inductance

Assume N coaxial circular filaments in free space, where filament i has a radius r_i, is placed at height h_i and carries a current I_i. The vector potential resulting from filament i is:

$$\boldsymbol{A}_i(\rho, z) = \frac{\mu}{2}\int_0^\infty dm\, r_i I_i J_1(mr_i) J_1(m\rho) e^{-m|z-h_i|} \boldsymbol{e}_\phi \qquad (26)$$

Analog to section 2.1, the mutual inductance between two coaxial circular filaments i and j is

$$M_{ij} = \frac{1}{I_i}\oint_{\partial C_j} \boldsymbol{A}_i . d\boldsymbol{l} = \pi\mu r_i r_j \int_0^\infty dm\, J_1(mr_i) J_1(mr_j) e^{-m|h_i-h_j|} \qquad (27)$$

The filaments can be bundled into a primary and a secondary side, denoted with indicies 1 and 2. The number of windings on the primary side is N_1, and $N_2 = N - N_1$ on the secondary side, see Figure 4. Moreover, the current through the primary coil is called I_1, and I_2 for the secondary coil.

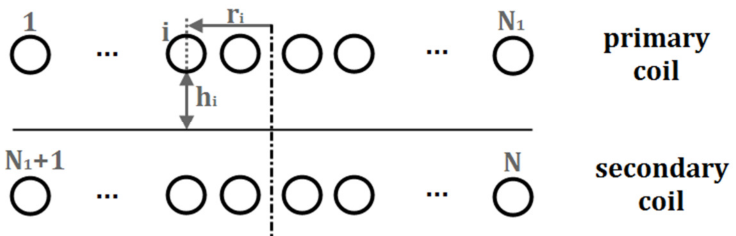

Figure 3: Configuration of the primary and secondary coil

The correct formula for the mutual inductance between the primary and secondary coil is

$$M = \sum_{j=1}^{N_1} \sum_{i=N_1+1}^{N} M_{ij} \qquad (28)$$

For coils with many turns and layers, calculating equation (28) is quite elaborate. According to [refD], the first order approximation of the mutual induction, M is

$$M \approx \pi\mu \bar{r}_1 \bar{r}_2 N_1 N_2 \int_0^\infty dm\, J_1(m\bar{r}_1) J_1(m\bar{r}_2) e^{-m|\bar{h}_1-\bar{h}_2|}, \qquad (29)$$

where $\bar{r}_1, \bar{r}_2, \bar{h}_1$ and $\bar{h}_2$ are the mean radii and heights of the coils, see equation (30), (31) and Figure 5.

$$\bar{r}_1 = \frac{1}{N_1}\sum_{i=1}^{N_1} r_i \qquad\qquad \bar{r}_1 = \frac{1}{N_2}\sum_{i=N_1+1}^{N} r_i \qquad (30)$$

$$\bar{h}_1 = \frac{1}{N_1}\sum_{i=1}^{N_1} h_i \qquad \bar{h}_1 = \frac{1}{N_2}\sum_{i=N_1+1}^{N} h_i \tag{31}$$

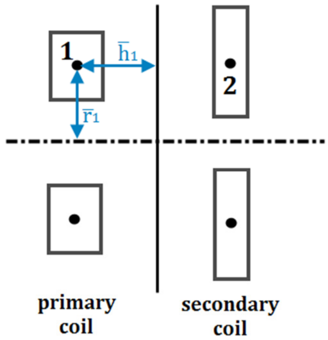

Figure 4: Cross section of the coils. Filaments for the first order approximation

5 Conclusion

In the previous sections, general expressions of mutual interaction and skin- & proximity losses for coaxial cylindrical coils were derived. Starting with simple coaxial circular filaments, the basic idea of mutual inductance and coupling factor was established. The extension to circular filaments with a finite cross section introduced skin- and proximity losses. With the principle of superposition, the mutual inductance for coaxial cylindrical coils was calculated. The main results of the theoretical model are summarized in Table 1.

The coupling factor *k* falls of linearly with the distance *d* for close coil separation.
$M \approx \pi\mu\bar{r}_1\bar{r}_2 N_1 N_2 \int_0^\infty dm\, J_1(m\bar{r}_1) J_1(m\bar{r}_2) e^{-m\lvert\bar{h}_1-\bar{h}_2\rvert}$
$R \approx \frac{2}{\mu\sigma\delta d}\left(L + \frac{I_2}{I_1}M\right)$

Table 1: Main results of the theoretical model for the mutual interaction between two coaxial cylindrical coils

Experimental treatments like [refA] show a good accuracy of these theoretical results. Furthermore, the coupling and skin & proximity losses are measured with the vector network analyzer Bode 100 in *"Measuring the Mutual Interaction between Coaxial Cylindrical Coils with the Bode 100"*, available on www.omicron-lab.com.

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Appendix A: Derivation of the vector potential

The diffusion equation

In homogeneous isotropic media with negligible free charges, Maxwell's equations (SI units, see [refB]) in the quasi-static limit are:

$$\nabla \wedge \boldsymbol{B} = \mu \boldsymbol{J} \tag{A.1}$$

$$\nabla \wedge \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial \mathrm{t}} \tag{A.2}$$

$$\nabla . \boldsymbol{B} = 0 \tag{A.3}$$

$$\nabla . \boldsymbol{E} = 0 \tag{A.4}$$

Here $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{E}$ the electric field, $\boldsymbol{J}$ the current density, μ the magnetic permeability and ε the electric permittivity. For conducting media, another useful relation is Ohm's law, where σ is the conductivity:

$$\boldsymbol{J} = \sigma \boldsymbol{E} \tag{A.5}$$

For non-conducting media, we will set $\sigma = 0$. Equations (A.1) to (A.5) fully describe the behavior of classical electromagnetic systems. Of course, appropriate boundary conditions are necessary to solve such partial differential equations. At an interface between two media, the boundary conditions are

$$\boldsymbol{n} . (\varepsilon_2 \boldsymbol{E}_2 - \varepsilon_1 \boldsymbol{E}_1) = \xi \tag{A.6}$$

$$\boldsymbol{n} \wedge (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0 \tag{A.7}$$

$$\boldsymbol{n} . (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \tag{A.8}$$

$$\boldsymbol{n} \wedge (\frac{\boldsymbol{B}_2}{\mu_2} - \frac{\boldsymbol{B}_1}{\mu_1}) = \boldsymbol{K} \tag{A.9}$$

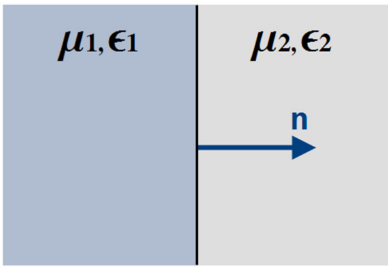

Figure A1: Boundary between two media

Note that $\boldsymbol{n}$ is a unit normal to the surface (Fig. A1), ξ a surface charge density and $\boldsymbol{K}$ a current on the surface plane. So far we have five differential equations and four boundary conditions which fully describe our system. Vector potentials will be a useful concept. A magnetic field $\boldsymbol{B}$ can be described by the curl of a vector potential $\boldsymbol{A}$, namely $\boldsymbol{B} = \boldsymbol{\nabla} \wedge \mathbf{A}$ with the choice of gauge freedom $\boldsymbol{\nabla} . \mathbf{A} = \mathbf{0}$ (Coulomb gauge). Hence we can write Faraday's law (A.2) as

$$\boldsymbol{\nabla} \wedge \left(\mathbf{E} + \frac{\partial \boldsymbol{A}}{\partial \mathrm{t}} \right) = \mathbf{0} \tag{A.10}$$

In general, the electric field is given by the vector potential $\boldsymbol{A}$ and a scalar potential φ:

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla} \varphi \tag{A.11}$$

For our calculation, we will neglect the scalar potential, i.e.$\varphi = 0$ for negligible free charges. Furthermore, we split the current density into two components, namely $\boldsymbol{J} = \boldsymbol{J}_{bound} + \boldsymbol{J}_{free}$, where $\boldsymbol{J}_{free}$, is a controllable current density in conductors and $\boldsymbol{J}_{bound}$ corresponds to a current density induced in a by Ampere's law. Rearranging Ampere's law (A.1) yields to

$$\nabla \wedge \boldsymbol{B} = \mu \boldsymbol{J}_{bound} + \mu \boldsymbol{J}_{free} = \mu \sigma \boldsymbol{E} + \mu \boldsymbol{J}_{free} \tag{A.12}$$

and inserting the electric field gives

$$\nabla \wedge \boldsymbol{B} = -\mu\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \mu \boldsymbol{J}_{free} \tag{A.13}$$

With the vector identity $\nabla \wedge (\nabla \wedge \boldsymbol{A}) = \nabla(\nabla.\boldsymbol{A}) - \nabla^2 \boldsymbol{A}$ and Coulomb gauge $\nabla.\boldsymbol{A} = 0$, we can write the diffusion equation as

$$\nabla^2 \boldsymbol{A} - \mu\sigma \frac{\partial \boldsymbol{A}}{\partial t} = -\mu \boldsymbol{J}_{free} \tag{A.14}$$

This differential equation will be the origin for further calculations. Note that $\nabla^2 \boldsymbol{A}$ is the vector Laplace operator of $\boldsymbol{A}$.

General symmetry arguments

For the further proceeding, mathematics is kept as simple as possible. Therefore, introducing a symmetry argument is an essential step. For circular coils, a model that obeys rotational symmetry is a valid approximation. It is advisable to describe the system in cylindrical coordinates, where the rotational invariant axis is set to be the horizontal component. Any position $\boldsymbol{x}$ in space can be described with a set of parameters $\rho \times \phi \times z \in \mathbb{R}_+ \times [0,2\pi) \times \mathbb{R}$ as:

$$\boldsymbol{x} = \begin{pmatrix} \rho \cos \phi \\ \rho \sin \phi \\ z \end{pmatrix} \tag{A.15}$$

The unit vectors are:

$$\boldsymbol{e}_\rho = \begin{pmatrix} \cos \phi \\ \sin \phi \\ z \end{pmatrix} \qquad \boldsymbol{e}_\theta = \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix} \qquad \boldsymbol{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{A.16}$$

In subsequent chapters, we will only consider circular currents, i.e. $\boldsymbol{J}(\boldsymbol{x},t) \equiv J(\boldsymbol{x},t)\boldsymbol{e}_\phi$ in conducting media. The direction of the vector potential $\boldsymbol{A}$ is determined by the current density $\boldsymbol{J}$. Ultimately, $\boldsymbol{A}(\boldsymbol{x},t) = A(\boldsymbol{x},t)\boldsymbol{e}_\phi$, where A is a scalar function. Furthermore, it is assumed that the electromagnetic fields and thus the vector potential will vary harmonically in time with a low angular frequency ω namely $\boldsymbol{A}(\boldsymbol{x},t) = A(\boldsymbol{x})\boldsymbol{e}_\phi e^{i\omega t}$. Finally, symmetry arguments and time dependence reduce the diffusion equation (A.14) for conducting media

$$\nabla^2 A - i\omega\mu\sigma A = -\mu J_{free}, \tag{A.17}$$

and non-conducting media

$$\nabla^2 A = 0. \tag{A.18}$$

Boundary conditions

By the use of symmetry arguments, it is also possible to simplify the boundary conditions (A.6) to (A.9). Therefore it is necessary to write the gradient operator in cylindrical coordinates:

$$\boldsymbol{\nabla} = \boldsymbol{e}_\rho \frac{\partial}{\partial \rho} + \boldsymbol{e}_\phi \frac{1}{\rho}\frac{\partial}{\partial x} + \boldsymbol{e}_z \frac{\partial}{\partial z} \tag{A.19}$$

From equation (A.11) and $\boldsymbol{B} = \boldsymbol{\nabla} \wedge \mathbf{A}$ it follows, that the components of the electromagnetic fields are

$$\boldsymbol{E} = -i\omega A \boldsymbol{e}_\phi \qquad \boldsymbol{B} = \boldsymbol{e}_z \frac{\partial A}{\partial x} - \boldsymbol{e}_\rho \frac{\partial A}{\partial x} \tag{A.20}$$

Since rotational symmetry along the z-axis is assumed, the unit normal to any surface, namely $\boldsymbol{n}$, can be written by a superposition of $\boldsymbol{n} = \alpha \boldsymbol{e}_\rho + \beta \boldsymbol{e}_z$, where $\alpha, \beta \in \mathbb{R}$ with the condition $\alpha^2 + \beta^2 = 1$. Furthermore, our model will only have vertical or horizontal boundaries such that either $\alpha = 1, \beta = 0$ or $\alpha = 0, \beta = 1$. The case of $\alpha = 1, \beta = 0$, is used for vertical boundary surfaces, e.g. for a cylindrical surface at radius r. Accordingly, the first boundary condition is always satisfied since $\boldsymbol{n} \,.\, \boldsymbol{E} = 0$ everywhere. The second boundary condition gives

$$\lim_{\rho \uparrow r} A = \lim_{\rho \downarrow r} A \tag{A.21}$$

Similarly, the third and fourth boundary conditions can be written as

$$\lim_{\rho \uparrow r} \frac{\partial A}{\partial z} = \lim_{\rho \downarrow r} \frac{\partial A}{\partial z} \tag{A.22}$$

$$\frac{1}{\mu_1} \lim_{\rho \uparrow r} \frac{\partial A}{\partial \rho} - \frac{1}{\mu_2} \lim_{\rho \downarrow r} \frac{\partial A}{\partial \rho} = K \tag{A.23}$$

where K is a scalar current density in the azimuthal direction. K is set to $K \equiv \left|\boldsymbol{J}_{free}\right|$. Now consider the conditions for a horizontal boundary at height h where $\alpha = 0, \beta = 1$.

$$\lim_{z \uparrow h} A = \lim_{z \downarrow h} A \tag{A.24}$$

$$\lim_{z \uparrow h} \frac{\partial A}{\partial \rho} = \lim_{z \downarrow h} \frac{\partial A}{\partial \rho} \tag{A.25}$$

$$-\frac{1}{\mu_1} \lim_{z \uparrow h} \frac{\partial A}{\partial z} + \frac{1}{\mu_2} \lim_{z \downarrow h} \frac{\partial A}{\partial z} = K \tag{A.26}$$

There are four more conditions that our electromagnetic fields should satisfy. They arise from the fact that we want to consider a physical system with a finite electromagnetic fields and dimensions. First, the vector potential has to be zero infinitely far away from the origin. Second, there should be no singularities in the vector potential within the region of interest. Consequently, both conditions can be specified in a mathematical way:

$$\lim_{\rho \to \infty} A = 0 \qquad \lim_{z \to \pm\infty} A = 0 \tag{A.27}$$

$$\forall R \in \mathbb{R}_+ : \ \left|\lim_{\rho \to R} A\right| < C, \text{ for a constant } C \in \mathbb{R}_+ \tag{A.28}$$

$$\forall Z \in \mathbb{R} \ : \ \left|\lim_{z \to Z} A\right| < D, \text{ for a constant } D \in \mathbb{R}_+ \tag{A.29}$$

The vector Laplace operator applied to the azimuthal component of the vector potential $\boldsymbol{A}$ gives

$$\boldsymbol{\nabla}^2 \boldsymbol{A} = \frac{\partial^2 A}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial A}{\partial \rho} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{\rho^2}, \tag{A.30}$$

where we have already neglected the ϕ-derivative. Thus the diffusion equation in a rotational invariant configuration for a harmonic vector field writes:

$$\frac{\partial^2 A}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial A}{\partial \rho} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{\rho^2} - i\omega\mu\sigma A = -\mu J_{free} \tag{A.31}$$

A general solution for the vector potential

In order to solve the diffusion equation (A.31) in a general way, we suggest using the technique of separating variables [refE]. A solution for the free space diffusion equation (A.18) is obtained below. For any separation variable $m \in \mathbb{R}$, the diffusion equation can be written as

$$\frac{\partial^2 A}{\partial z^2} = m^2 A \tag{A.32}$$

$$\frac{\partial^2 A}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial A}{\partial \rho} - \frac{A}{\rho^2} = -m^2 A \tag{A.33}$$

Using an approach of $A = F(\rho)G(z)$, gives

$$\frac{\partial^2 G}{\partial z^2} - m^2 G = 0 \tag{A.34}$$

$$\frac{\partial^2 A}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial A}{\partial \rho} + (m^2 - \frac{1}{\rho^2})A = 0 \tag{A.35}$$

For equation (A.34) the solution is a linear combination of exponential functions, namely

$$G(z) = \tilde{A}(m)e^{m|z|} + \tilde{B}(m)e^{-m|z|}, \tag{A.36}$$

and for equation (A.35), a combination of Bessel- and Neumann-functions is appropriate

$$F(\rho) = \tilde{C}(m)J_1(m\rho) + \tilde{D}(m)N_1(m\rho) \tag{A.37}$$

Note that $\tilde{A}(m)$, $\tilde{B}(m)$, $\tilde{C}(m)$ and $\tilde{D}(m)$ are coefficients dependent on m, J_1 is the first kind Bessel-function of first order and N_1 the first kind Neumann-function of first order. These coefficients are determined by the boundary conditions. The total solution for the vector potential can be written as

$$A = \int_0^\infty dm\,\left[\tilde{A}(m)e^{m|z|} + \tilde{B}(m)e^{-m|z|}\right]\left[\tilde{C}(m)J_1(m\rho) + \tilde{D}(m)N_1(m\rho)\right] \tag{A.38}$$

Appendix B: Vector potentials of coaxial circular filaments

One circular filament

Place a simple circular filament with radius r in free space. The filament should carry a sinusoidal current, described by the current density $\boldsymbol{J}_{free}$.

$$\boldsymbol{J}_{free} = I\delta(z)\frac{\delta(\rho - r)}{r}\boldsymbol{e}_\phi \tag{A.39}$$

Split the region of interest into two parts I and II, such that the circular filament lies on the boundary plane. The general integral solution for physical systems with a limited $\boldsymbol{A}$ is

$$A_{\mathrm{I}} = \int_0^\infty dm\tilde{B}_{\mathrm{I}}e^{-mz}J_1(m\rho) \tag{A.40}$$

$$A_{\mathrm{II}} = \int_0^\infty dm\tilde{B}_{\mathrm{II}}e^{mz}J_1(m\rho) \tag{A.41}$$

The coefficients $\tilde{C}(m)$ from the first order Bessel function are collectively absorbed in $\tilde{B}_{\mathrm{I}}$ and $\tilde{B}_{\mathrm{II}}$. Applying the first boundary condition at $z = 0$ for the vector potential gives

$$\int_0^\infty dm\tilde{B}_\text{I} e^{-mz} J_1(m\rho) = \int_0^\infty dm\tilde{B}_\text{II} e^{mz} J_1(m\rho) \tag{A.42}$$

Now multiply both sides by the integral operator $\int_0^\infty \rho d\rho J_1(m'\rho)$ and use the Fourier-Bessel identity $\int_0^\infty \rho d\rho J_1(m'\rho) J_1(m\rho) = \frac{\delta(m'-m)}{m}$. This yields to

$$\tilde{B}_\text{I} = \tilde{B}_\text{II}. \tag{A.43}$$

The relation between the loop current, that forces a change in the magnetic field, and thus the change in the vector potential has to be considered. The magnetic permeability is the same in all regions, i.e. $\mu_1 = \mu_2 = \mu$.

$$-\frac{\partial A_\text{I}}{\partial z}\bigg|_{z=0} + \frac{\partial A_\text{II}}{\partial z}\bigg|_{z=0} = \mu I \delta(\rho - r) \tag{A.44}$$

$$\tilde{B}_\text{I} + \tilde{B}_\text{II} = \mu r I J_1(mr) \tag{A.45}$$

$$A = \frac{\mu}{2}\int_0^\infty dm r I J_1(mr) J_1(m\rho) \tag{A.46}$$

Two coaxial circular filaments

Let us now consider two coaxial circular with radii r_1 and r_2, separated by a height h, in free space (see Figure 1). Both carry a sinusoidal current, represented by a current density $\boldsymbol{J}_{free1}$ and $\boldsymbol{J}_{free2}$.

$$\boldsymbol{J}_{free1} = I_1 \delta(z) \frac{\delta(\rho - r_1)}{r_1} \boldsymbol{e}_\phi \tag{A.47}$$

$$\boldsymbol{J}_{free2} = I_2 \delta(z-h) \frac{\delta(\rho - r_2)}{r_2} \boldsymbol{e}_\phi \tag{A.48}$$

According to Figure 1, we can split or model into three regions of interest.

$$A_\text{I} = \int_0^\infty dm\tilde{B}_\text{I} e^{-mz} J_1(m\rho) \qquad \text{for } z > h \tag{A.49}$$

$$A_\text{II} = \int_0^\infty dm\left[\tilde{A}_\text{II} e^{mz} + \tilde{B}_\text{II} e^{-mz}\right] J_1(m\rho) \qquad 0 < z < h \tag{A.50}$$

$$A_\text{III} = \int_0^\infty dm\tilde{B}_\text{III} e^{mz} J_1(m\rho) \qquad z > h \tag{A.51}$$

In this constellation, there are two horizontal boundaries. Applying the first boundary condition for the vector potential and applying the Fourier-Bessel identity gives

$$\tilde{B}_\text{I} e^{-mh} = \tilde{A}_\text{II} e^{mh} + \tilde{B}_\text{II} e^{-mh}. \tag{A.52}$$

Similarly, the second horizontal boundary at $z = 0$ leads to

$$\tilde{B}_\text{III} = \tilde{A}_\text{II} + \tilde{B}_\text{II}. \tag{A.53}$$

Regarding boundary condition (A.26) with $\mu_1 = \mu_2 = \mu_3 = \mu$ gives

$$-\frac{\partial A_\text{III}}{\partial z}\bigg|_{z=0} + \frac{\partial A_\text{II}}{\partial z}\bigg|_{z=0} = \mu I_1 \delta(\rho - r_1), \tag{A.54}$$

and

$$-\left.\frac{\partial A_{\mathrm{II}}}{\partial z}\right|_{z=h}+\left.\frac{\partial A_{\mathrm{I}}}{\partial z}\right|_{z=h}=\mu I_1\delta(\rho-r_2), \tag{A.55}$$

which leads to

$$-\tilde{A}_{\mathrm{II}}+\tilde{B}_{\mathrm{II}}+\tilde{B}_{\mathrm{III}}=\mu r_1 I_1 J_1(m r_1), \tag{A.56}$$

and

$$\tilde{B}_{\mathrm{I}}e^{-mh}+\tilde{A}_{\mathrm{II}}e^{mh}-\tilde{B}_{\mathrm{II}}e^{-mh}=\mu r_2 I_2 J_1(m r_2). \tag{A.57}$$

Algebraic manipulations of (A.56) and (A.57) yield to

$$2\tilde{A}_{\mathrm{II}}e^{mh}=\mu r_2 I_2 J_1(m r_2), \tag{A.58}$$

and

$$2\tilde{B}_{\mathrm{II}}=\mu r_2 I_2 J_1(m r_2). \tag{A.59}$$

The total vector potential for any point (ρ, z) is thus given by

$$\boldsymbol{A}(\rho,z)=\frac{\mu}{2}\int_0^\infty dm\left[r_1 I_1 J_1(m r_1)e^{-m|z|}+r_2 I_2 J_1(m r_2)e^{-m|z-h|}\right]J_1(m\rho)\boldsymbol{e}_\phi. \tag{A.60}$$

Although we don't give an explicit prove that the integral expressions for $\boldsymbol{A}$ are convergent, one can see that (A.60) is a superposition of two single filaments, each with a vector potential equivalent to (A.46).